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Topology Aware Link Throughput of Slotted Aloha in Rayleigh Block Fading Channels

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***Abstract*—In this work, we present an accurate analysis of the probability of successful transmission in a slotted Aloha network with an arbitrary topology, provided that the channel can be accurately modeled as Rayleigh block fading channels. The obtained expression also takes into account the effect of different physical layer parameters such as modulation and coding methods. However, its computational complexity grows quickly as the network size increases. To address this, we also present an accurate approximation method in which the probability of success for a link is predicted by considering only a subset of the interfering nodes. A sufficient condition for the accuracy of this prediction is also presented. The validity of the proposed methods are verified by a series of simulations.**

I. Introduction

Despite the fact that it has been four decades since the Aloha protocol was first proposed by Abramson [1], different features of this protocol and its variants are still being actively studied by the research community. This is partly due to the fact that this family of medium access control (MAC) protocols is still an attractive choice for many applications due to its simplicity, its ability to operate without centralized coordination, and its adaptability to bursty traffic. In addition, existing solutions tend to consider different sets of assumptions depending on the specifics of the considered application or the aspects of the problem they find interesting.

In this work, the slotted Aloha protocol is considered. We present an accurate analysis of the probability of successful transmission of a link in networks in which the channels can be accurately modeled as Rayleigh block fading channels. The method presented here makes no restricting assumption on the distributions of nodes or the size of the network, which differentiates it from various existing analyses. Many of the existing results, for instance, require a large network size in which nodes are distributed according to Poisson distribution [2]–[4]. Another common assumption not required this work, is the premise that all nodes transmit with the same probability of transmission [2]–[4]. In this aspect, our work is similar to [5] in which nodes are allowed to have different probabilities of transmission. In addition, similar to [6], [7] the effect of physical layer parameters such as modulation, coding methods, and packet size on the probability of success is considered.

However, our method has the drawback of quickly becoming computationally demanding as the number of nodes in the network increases. This problem is common among methods which consider the detailed state of nodes in the network. For instance in [5], slotted Aloha networks are modeled by a Markov model where the number of states is exponential in the number of nodes. Similarly, the complexity of the method presented in [8] for the framed slotted Aloha is shown to grow very fast with the number of nodes.

To address the complexity problem, we develop an approximation method which requires considerably less amount of computation. We initially estimate the probability of success by only considering a subset of interfering nodes. We further demonstrate that if this subset is chosen such that a necessary condition is satisfied, the error caused by this approximation can be predicted with high reliability as a function of the interfering powers and the transmission probabilities of the removed nodes. Therefore, as confirmed by the simulation results in Sec. V, the approximation error can be further reduced.

It should be mentioned that the derived approximation is also applicable to non-Aloha time-slotted networks (as long as the assumptions in Sections II–IV are satisfied).

The rest of this paper is organized as follows. In Sec. II, the interference and network model are defined. A detailed and partly novel analysis of probability of success is presented Sec. III. The main contribution of the paper, the derivation of a low-complex approximation method for the probability of success, is found in Sec. IV. The validity of this approximation is examined by a series of simulations in Sec. V. The paper is concluded in Sec. VI.

II. System model

An ad hoc network with N nodes is considered. These nodes are indexed by integers $1, 2, \ldots, N$ and communicating using the slotted Aloha MAC protocol. The probability of node i transmitting in a randomly chosen time-slot is denoted by p_i. It is assumed that nodes are synchronized and the entire time-slot is utilized to transmit a packet. Thus, in the case of a packet collision, packets are fully overlapped.

The instantaneous received power from node i at node j is represented by $P_{i,j}$ and is given by $P_{i,j} = \kappa_{i,j}\bar{P}_{i,j}$, where $\bar{P}_{i,j}$ is the average received power from node i at node j and $\kappa_{i,j}$ models the effect of small-scale fading on the instantaneous received signal power. The level of mobility of nodes and the environment are assumed to be such that the small-scale fading can be modeled as block fading over a single time slot. Since collisions are assumed to be fully overlapping, the channel can be modeled as a block interference channel [9]. The small scale fading is assumed to be Rayleigh distributed, hence $\kappa_{i,j}$ is a unit mean exponentially distributed random variable. The effects of path loss and shadowing are captured by $\bar{P}_{i,j}$ which is assumed to be slowly varying.

A communication link is considered where t and r represent a transmitter and a receiver respectively. For this link, an interfering node is defined as any node other than t and r which is transmitting in the same time slot. If set I contains the index of all the interfering nodes in a given time slot, the instantaneous signal-to-interference-and-noise-ratio (SINR) for t, r, and a given the set I can be written as

$$\gamma(t,r|I) = \frac{\kappa_{t,r}\bar{P}_{t,r}}{P_n + \sum_{i\in I}\kappa_{i,r}L_{i,r}\bar{P}_{i,r}}, \tag{1}$$

where P_n is the noise power and $L_{i,r}$ is the processing gain which depends on correlation properties of the transmitted signal from node i and the receiving filter at r. In absence of spread spectrum modulations, $L_{i,r}$ is often assumed to be one, which is also adopted in this paper. However, the method outlined here can be easily extended to case where spread spectrum modulation is used.

III. Probability of Successful Transmission

Let us define a set T as a set of all the node indices excluding the indices of the transmitter and the receiver as follows

$$T \triangleq \{1,\ldots,N\} \setminus \{t,r\}, \tag{2}$$

where $\setminus$ denotes set subtraction. Given a set of interfering nodes, $I \subset T$, a packet transmitted by t is assumed to be successfully received at r if the following events occurs simultaneously,

$$A \triangleq \{r \text{ is not transmitting}\}$$
$$B \triangleq \{r \text{ locks to the packet transmitted by } t|I\}$$
$$C \triangleq \{\text{The preamble and header is decoded successfully}|I\}$$
$$D \triangleq \{\text{The payload is decoded successfully}|I\}.$$

Thus, the conditional probability of successful transmission is given by

$$P_s(t,r,T|I) = \Pr\left(A\cap B\cap C\cap D\right), \tag{3}$$

where $P_s(t,r,T|I)$ is the conditional probability of successful transmission between t and r for a set of possible interfering node indices T and given the currently active interferers in I. Solving (3) as an intersection of these events is not trivial. Instead, using the conditional probability law, (3) is rewritten as follows

$$P_s(t,r,T|I) = \Pr\left(C|A\cap B\cap D\right)\Pr\left(B\cap D|A\right)\Pr\left(A\right), \tag{4}$$

where each term can be easily calculated. Starting from left, the first term in (4) is the probability that node r is not transmitting and is given by

$$\Pr\left(A\right) = 1 - p_r. \tag{5}$$

To evaluate $\Pr\left(B\cap D|A\right)$, events B and D must be described in more details. Starting with the event B, we need to define the conditions for a receiver to successfully lock to a specific packet in the presence of interference. While, in general, the required conditions are hardware dependent, in order to not limit our analysis to a particular hardware, we assume that the receiver always locks to the packet with the highest received power. This assumption is reasonable in a synchronized time slotted system such as slotted Aloha considered in this work. Therefore, event B can be rewritten as follow

$$\begin{aligned} B &= \{r \text{ locks to the packet transmitted by } t|I\} \\ &= \{P_{t,r} > P_{i,r}, \forall i \in I\} \\ &= \left\{\kappa_{t,r} > \kappa_{i,r}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}, \forall i\in I\right\}. \end{aligned} \tag{6}$$

The event D which is the probability of successful reception of the payload, is equal to the block error probability when the payload contains a single block of coded bits. It is shown in [7] that the block error probability in Rayleigh block interference channel can be accurately approximated by applying the threshold method to $\gamma(t,r|I)$. Therefore, assuming that the payload contains a single block of coded bits, the event D can be rewritten as follow

$$\begin{aligned} D &= \{\text{The payload is decoded successfully}|I\} \\ &\simeq \{\gamma(t,r|I) > \Theta\} \\ &= \left\{\kappa_{t,r} > \Theta\left(\frac{P_n}{\bar{P}_{t,r}} + \sum_{i\in I}\kappa_{i,r}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}\right)\right\}, \end{aligned} \tag{7}$$

where $L_{i,r}$ are assumed to be 1 and $\Theta \geq 1$ is the model parameter which depends on the parameters of the physical layer such as modulation, coding methods, and block size [10].

A careful examination of (6) and (7) reveals that D implies B, e.g., $D \subset B$, and we have

$$\begin{aligned} \Pr\left(B\cap D|A\right) &= \Pr(B\cap D) \\ &= \Pr(D) \\ &= \Pr\left\{\kappa_{t,r} > \Theta\left(\frac{P_n}{\bar{P}_{t,r}} + \sum_{i\in I}\kappa_{i,r}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}\right)\right\} \\ &= \frac{\exp(-\Theta P_n/\bar{P}_{t,r})}{\prod_{i\in I}(1+\Theta P_{i,r}/P_{t,r})}, \end{aligned} \tag{8}$$

where the first line is the result of both B and D being independent from A.

The first factor in (4) is the probability that the preamble and header are successfully received given that the events A, B and D have occurred (i.e. $\Pr\left(C|A\cap B\cap D\right)$). The condition D implies that $\gamma(t,r|I)$ has been sufficiently high for the payload to be received successfully. Since in many practical systems, the header and preamble are often designed to withstand higher interference than the payload, we assume that $\Pr\left(C|A\cap B\cap D\right) \simeq 1$. In general, however, $P_s(t,r,T|I)$ obtained based on this assumption provides an upper bound to the probability of successful transmission.

Combining (5), (8), and the assumption that $\Pr(C|A \cap B \cap D) = 1$, the conditional probability of successful transmission is given by

$$P_s(t,r,T|I) = \frac{(1-p_r)\exp(-\Theta P_n/\bar{P}_{t,r})}{\prod_{i\in I}(1+\Theta\bar{P}_{i,r}/\bar{P}_{t,r})}. \qquad (9)$$

Consequently, the probability of successful transmission can be obtained by averaging (9) over all possible set of interfering nodes. That is

$$P_s(t,r,T) = \sum_{I\subset T} P_s(t,r,T|I)Q(T,I), \qquad (10)$$

where $Q(T,I)$ is the probability the nodes in $I \subset T$ are transmitting, i.e.,

$$Q(Y,X) \triangleq \prod_{i\in X} p_i \prod_{j\in Y\setminus X}(1-p_j). \qquad (11)$$

For (11) to make sense, we insist on $X \subset Y$. Moreover, we note that

$$\sum_{I\subset T} Q(T,I) = 1. \qquad (12)$$

The main challenge in using the result in (10) in a large network is that the number of possible interfering sets grows extremely fast with N. In the next section, a significantly less complex and yet accurate approximation to (10) is obtained.

IV. Approximation

The computational complexity of (10) becomes quickly prohibitive as a function of number of nodes in the network due to the drastic increase in the number of subsets of T which is given by

$$|\{I|I\subset T\}| = \sum_{i=0}^{N-2}\binom{N-2}{i}, \qquad (13)$$

where $|X|$ here denotes the cardinality of the set X. This problem can be diminished if the number of subsets of T is virtually reduced by removing less significant interfering nodes from T. Reducing the number of interfering nodes, evidently results in an upper bound for the probability of success. One difficulty with this approach, however, is that it is not clear what is the best way to choose these less significant nodes. As an example, consider two interfering nodes, indexed by i and j, such that $p_i > p_j$ and $\bar{P}_{i,r} < \bar{P}_{j,r}$. In this case, while node i transmit more often than node j, the interference it causes is less serious. Therefore, a method of scoring nodes based on their relevance to the accuracy of this approximation is required. In addition, it is not clear how many interfering nodes can be ignored such that this approximation remains accurate. In the rest of this section, we aim to address these issues by finding a relation between the probability of success in an entire network and the probability of success obtained over a reduced set of interfering nodes.

Let $\tilde{T}$ represents the set of removed interfering node indices. A reduced set of interfering node indices is then defined as

$$\hat{T} \triangleq T\setminus\tilde{T}. \qquad (14)$$

Similarly, for a given set of interfering nodes indices I, we define $\hat{I}$, and $\tilde{I}$ as

$$\hat{I} \triangleq I\cap\hat{T} \qquad (15)$$

$$\tilde{I} \triangleq I\cap\tilde{T} = I\setminus\hat{I}. \qquad (16)$$

Using the result in (9), it is easy to show that $P_s(t,r,T|I)$ can be written as

$$P_s(t,r,T|I) = \frac{P_s(t,r,\hat{T}|\hat{I})}{1+\zeta(\tilde{I})}, \qquad (17)$$

where $\zeta(\tilde{I})$ is given by

$$\zeta(\tilde{I}) = -1 + \prod_{i\in\tilde{I}}\left(1+\Theta\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}\right). \qquad (18)$$

Assuming that the removed nodes are chosen such that $\bar{P}_{i,r}/\bar{P}_{t,r} \ll 1$ for all $i \in \tilde{T}$, all the terms of the form $\bar{P}_{i,r}\bar{P}_{j,r}/\bar{P}^2_{t,r}$ can be safely ignored for all $i,j \in \tilde{T}$. Thus, $\zeta(\tilde{I})$ can be approximated by $\widehat{\zeta}(\tilde{I})$ defined as follows

$$\widehat{\zeta}(\tilde{I}) \triangleq \Theta\sum_{i\in\tilde{I}}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}. \qquad (19)$$

Furthermore, if $\zeta(\tilde{I}) \ll 1$, the (17) can be further simplified using the Taylor expansion as

$$P_s(t,r,T|I) \simeq P_s(t,r,\hat{T}|\hat{I})\left(1-\Theta\sum_{i\in\tilde{I}}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}\right). \qquad (20)$$

The expression in (20) demonstrates how the conditional probability of success can be accurately predicted in terms of the conditional probability of success which is obtained over a reduced set of interferers (i.e., $\hat{I}$ and $\hat{T}$). The sufficient condition for the accuracy of (20) is that the removed nodes satisfy $\zeta(\tilde{I}) \ll 1$.

In the following, we utilize (20) to obtain a similar relation between $P_s(t,r,T)$ and $P_s(t,r,\hat{T})$. We start by dividing the summation in (10) into smaller sums as follows

$$\begin{aligned} P_s(t,r,T) &= \sum_{I\subset T} P_s(t,r,T|I)Q(T,I) \\ &= \sum_{\tilde{I}\subset\tilde{T}}\sum_{\substack{I\subset T\\ I\cap\tilde{T}=\tilde{I}}} P_s(t,r,T|I)Q(T,I). \qquad (21)\end{aligned}$$

Each of the inner sums in (21) can be written in the terms of the reduced set of interfering nodes, $\hat{T}$, as

$$\begin{aligned} &P_s(t,r,T) \\ &\simeq \sum_{\tilde{I}\subset\tilde{T}}\sum_{\substack{I\subset T\\ I\cap\tilde{T}=\tilde{I}}} P_s(t,r,\hat{T}|\hat{I})(1-\zeta(\tilde{I}))Q(\hat{T},\hat{I})Q(\tilde{T},\tilde{I}) \qquad (22) \\ &= \sum_{\tilde{I}\subset\tilde{T}}\left(1-\zeta(\tilde{I})\right)Q(\tilde{T},\tilde{I})\sum_{\substack{I\subset T\\ I\cap\tilde{T}=\tilde{I}}} P_s(t,r,\hat{T}|\hat{I})Q(\hat{T},\hat{I}), \\ &\qquad (23)\end{aligned}$$

where the approximation in (22) is obtained by applying the Taylor expansion to (17) given $\zeta(\tilde{I}) \ll 1$. The validity of the relation between $Q(T,I)$, $Q(\hat{T},\hat{I})$, and $Q(\tilde{T},\tilde{I})$ is proven in the appendix.

The inner summation in (23), sums over all subsets of T such that $\tilde{I}$ is fixed. Knowing $I=\tilde{I}\cup\hat{I}$, the summation can be equivalently performed over all $\hat{I}\subset\hat{T}$ for the fixed $\tilde{I}$. However, since the terms in this sum only depend on $\hat{T}$ and $\hat{I}$, the condition on fixed $\tilde{I}$ has no effect. Therefore, the right-hand side of (23) can be simplified as

$$\begin{aligned}
&\sum_{\tilde{I}\subset\tilde{T}}\left(1-\zeta(\tilde{I})\right)Q(\tilde{T},\tilde{I})\sum_{\substack{I\subset T\\ I\cap\tilde{T}=\tilde{I}}}P_s(t,r,\hat{T}|\hat{I})Q(\hat{T},\hat{I})\\
&=\sum_{\tilde{I}\subset\tilde{T}}\left(1-\zeta(\tilde{I})\right)Q(\tilde{T},\tilde{I})\sum_{\substack{\hat{I}\subset\hat{T}\\ I\cap\tilde{T}=\tilde{I}}}P_s(t,r,\hat{T}|\hat{I})Q(\hat{T},\hat{I})\\
&=\sum_{\tilde{I}\subset\tilde{T}}\left(1-\zeta(\tilde{I})\right)Q(\tilde{T},\tilde{I})\sum_{\hat{I}\subset\hat{T}}P_s(t,r,\hat{T}|\hat{I})Q(\hat{T},\hat{I})\\
&=P_s(t,r,\hat{T})\sum_{\tilde{I}\subset\tilde{T}}\left(1-\zeta(\tilde{I})\right)Q(\tilde{T},\tilde{I}). \qquad (24)
\end{aligned}$$

The summation in (24) can be further simplified as follows

$$\begin{aligned}
\sum_{\tilde{I}\subset\tilde{T}}\left(1-\zeta(\tilde{I})\right)Q(\tilde{T},\tilde{I})&=\sum_{\tilde{I}\subset\tilde{T}}Q(\tilde{T},\tilde{I})-\sum_{\tilde{I}\subset\tilde{T}}\zeta(\tilde{I})Q(\tilde{T},\tilde{I})\\
&=1-\sum_{\tilde{I}\subset\tilde{T}}\zeta(\tilde{I})Q(\tilde{T},\tilde{I})\\
&\simeq 1-\sum_{\tilde{I}\subset\tilde{T}}\sum_{i\in\tilde{I}}\Theta\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}Q(\tilde{T},\tilde{I}),
\end{aligned} \qquad (25)$$

where the approximation in (25) is obtained by replacing $\zeta(\tilde{I})$ with $\widehat{\zeta}(\tilde{I})$ given in (19). The summations in the right-hand side of (25) can be rewritten as

$$\begin{aligned}
&\sum_{\tilde{I}\subset\tilde{T}}\sum_{i\in\tilde{I}}\Theta\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}Q(\tilde{T},\tilde{I}) && (26)\\
&=\Theta\sum_{i\in\tilde{T}}\sum_{\substack{\tilde{I}\subset\tilde{T}\\ i\in\tilde{I}}}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}Q(\tilde{T},\tilde{I}) && (27)\\
&=\Theta\sum_{i\in\tilde{T}}\sum_{\substack{\tilde{I}\subset\tilde{T}\\ i\in\tilde{I}}}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}p_iQ(\tilde{T}\setminus\{i\},\tilde{I}\setminus\{i\}) && (28)\\
&=\Theta\sum_{i\in\tilde{T}}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}p_i\sum_{\substack{\tilde{I}\subset\tilde{T}\\ i\in\tilde{I}}}Q(\tilde{T}\setminus\{i\},\tilde{I}\setminus\{i\}), && (29)
\end{aligned}$$

where (27) is obtained by carefully exchanging the inner and the outer summation in (26). The validity of (28) is shown in the appendix. The inner sum in (29) is a summation over the probability of transmission events for all subsets of $\tilde{T}\setminus\{i\}$ for a fixed node i which implies that this sums is one for all $i\in\tilde{T}$. That is

$$\sum_{\substack{\tilde{I}\subset\tilde{T}\\ i\in\tilde{I}}}Q(\tilde{T}\setminus\{i\},\tilde{I}\setminus\{i\})=\sum_{I^*\subset T^*}Q(T^*,I^*)=1, \qquad (30)$$

where $T^*=\tilde{T}\setminus\{i\}$. Combining (24), (25), (29), and (30), we have

$$P_s(t,r,T)\simeq P_s(t,r,\hat{T})\left(1-\Theta\sum_{i\in\tilde{T}}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}p_i\right). \qquad (31)$$

The expression in (31) notably provides a method of obtaining the probability of success for the entire network by adjusting an approximation of probability of success obtained over a reduced set of nodes, given $\zeta(\tilde{I})\ll 1$ for all $\tilde{I}\subset\tilde{T}$. Therefore, we define the prediction of $P_s(t,r,T)$ as

$$\widehat{P}_s(t,r,T,\hat{T})\triangleq P_s(t,r,\hat{T})\left(1-\Theta\sum_{i\in\tilde{T}}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}p_i\right) \qquad (32)$$

which is accurate if

$$\Delta\triangleq\Theta\sum_{i\in\tilde{T}}\frac{\bar{P}_{i,r}}{\bar{P}_{t,r}}\ll 1. \qquad (33)$$

The condition in (33) sets some constrains on how the removed nodes should be chosen. To ensure that Δ remains significantly less than 1 for as many removed nodes as possible, the nodes with the lowest interfering power should be removed first. In addition, (33) also implies that for a randomly chosen link and a fixed number of removed nodes, the condition on Δ is more likely to be satisfied in topologies with a low node density.

It also worth noting that the network load (i.e., the transmission probabilities of nodes) has no effect on Δ. Nevertheless, the error contribution to the probability of success depends on both the ratio of the powers and the probability of transmission.

V. Simulations

A slotted Aloha network is considered in which nodes are randomly distributed over a square area with the edge length L m which is chosen such that the node density is λ (i.e., $L=\sqrt{N/\lambda}$). To simulate unequal probability of transmission for nodes in the network, p_i is randomly chosen between $p_{\text{min}}=0.01$ and $p_{\text{max}}=0.1$ for each of the simulated topologies. To simplify the simulations, shadow fading is ignored and the average received power is modeled by log-distance path-loss model as follow

$$\bar{P}_{i,k}=P_0\left(\frac{d_0}{d_{i,k}}\right)^{\alpha}, \qquad (34)$$

where $d_{i,k}$ is the distance between node i and node k and P_0 is the average received power at distance d_0. In our simulations, $P_0/P_n=10$ dB at reference distance $d_0=1$m and the path-loss exponent is $\alpha=4$. The information bits are assumed to be coded using a simple rate $1/2$ convolutional code where its generator polynomial is $(23,35)$ in octal format. The packets only contain a single payload of 500 bytes of coded bits and are modulated with the BPSK signaling. These parameters and other simulation parameters are summarized in Table I.

To compile a list of potential receivers for each node, the maximum transmission range of nodes need to be known.

TABLE I
SIMULATION PARAMETERS

p_{min}	0.01	R	2.18 m
p_{max}	0.1	Modulation	BPSK
P_0/P_n	10 dB	Generator polynomial	$(23, 35)$
d_0	1 m	Free distance	7
α	4	Code rate	$1/2$
Θ	3.1 dB	Block size	500 byte
μ	0.99		

Therefore, we define the maximum transmission range (R) of a node as the range around a transmitter within which the probability of successful reception in absence of interference is above a threshold, μ. Combining (34) and (9), it is easy to show that R is given by

$$R = d_0 \sqrt[\alpha]{\frac{-P_0 \ln \mu}{\Theta P_n}}. \tag{35}$$

Thus, the potential receivers for a node are defined as all nodes which are within distance R from it. In our simulation, only connected networks are considered in which all nodes have at least a single potential receiver.

In the first set of experiments, we compare the accuracy of $\widehat{P}_s(t, r, T, \hat{T})$, which compensates for the error, with $P_s(t, r, \hat{T})$, which simply ignores the effect of nodes in $\tilde{T}$. The numerical results are obtained for 1000 randomly generated networks of 20 nodes. For a network of this size, the cardinality S is small enough to allow obtaining $P_s(t, r, T)$ from (10). The accuracy is measured in terms of the relative prediction error, $\xi(x)$, defined as follow

$$\xi(x) \triangleq \frac{|P_s(t, r, T) - x|}{P_s(t, r, T)}, \tag{36}$$

where $|.|$ here denotes absolute value. The cumulative distribution functions (cdf) of $\xi\left(\widehat{P}_s(t, r, T, \hat{T})\right)$ and $\xi\left(P_s(t, r, \hat{T})\right)$ for $\hat{T}$ containing 5, 10 and 15 nodes with the highest interfering power are shown in Fig. 1. Comparing the curves with the same $\hat{T}$ confirms that the prediction proposed in (32) significantly improves the accuracy. Note that the results in this figure are obtained without checking if condition (33) is satisfied (i.e., $\Delta \ll 1$).

We further examine the accuracy of $\widehat{P}_s(t, r, T, \hat{T})$ in a large network of $N = 100$ nodes. For a network of this size, obtaining $P_s(t, r, T)$ from (10) is no longer practical. Therefore, $P_s(t, r, T)$ is obtained from simulating 1000 randomly generated networks. For a randomly selected link, $P_s(t, r, T)$ is estimated by $K = 1,000,000$ iterations. In each iteration, a set of interfering nodes (i.e., I) is randomly generated in accordance with the transmission probabilities of the nodes. The outcome of trial k is a binary process, χ_k, that is defined as follows

$$\chi_k = \begin{cases} 1, & \gamma(t, r|I) \geq \Theta \\ 0, & \text{otherwise} \end{cases} \tag{37}$$

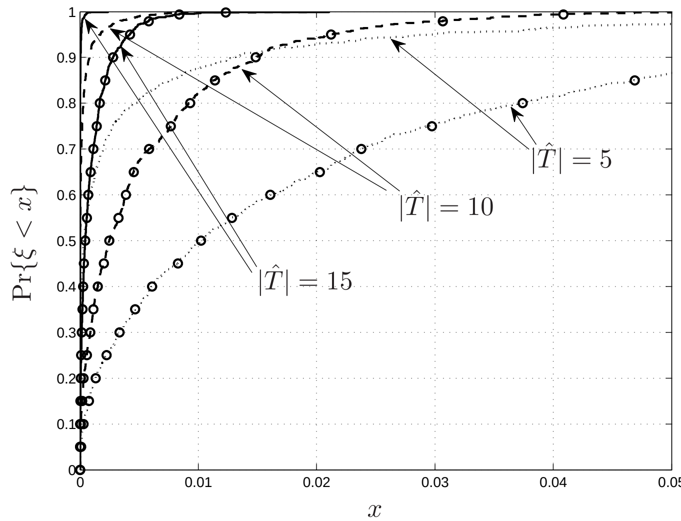

Fig. 1. The cdfs of $\xi\left(\widehat{P}_s(t, r, T, \hat{T})\right)$ for $|\hat{T}| = 15$ (solid line), $|\hat{T}| = 10$ (dashed line), and $|\hat{T}| = 5$ (dotted line). Also, the cdfs of $\xi\left(P_s(t, r, \hat{T})\right)$ for $|\hat{T}| = 15$ (solid line and circle), $|\hat{T}| = 10$ (dashed line and circle) and $|\hat{T}| = 5$ (dotted line and circle); $N = 20$, $\lambda = 5$.

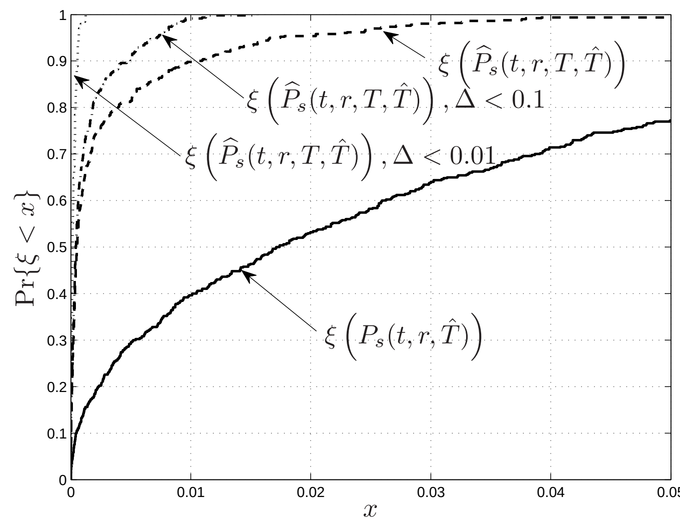

Fig. 2. The cdf of $\xi\left(P_s(t, r, \hat{T})\right)$ (solid line) and the cdfs of $\xi\left(\widehat{P}_s(t, r, T, \hat{T})\right)$ for all Δ (dashed line), for $\Delta < 0.1$ (dashed and dotted line), and for $\Delta < 0.01$ (dotted line); $N = 100$, $\lambda = 10$.

The probability of successful transmission for this link is then estimated by

$$P_s(t, r, T) \simeq \frac{1}{K} \sum_{k=1}^{K} \chi_k. \tag{38}$$

For the same topology, transmitter, and receiver, $\widehat{P}_s(t, r, T, \hat{T})$ and $P_s(t, r, \hat{T})$ are also obtained for a $\hat{T}$ containing only 20 nodes with the highest interfering power. The cdf of resulting relative errors are shown in Fig. 2. In addition, the cdfs of $\xi(\widehat{P}_s(t, r, T, \hat{T}))$, obtained by only considering the links for which $\Delta < 0.1$ and $\Delta < 0.01$, are also shown in Fig. 2. It can be seen that, as expected, the

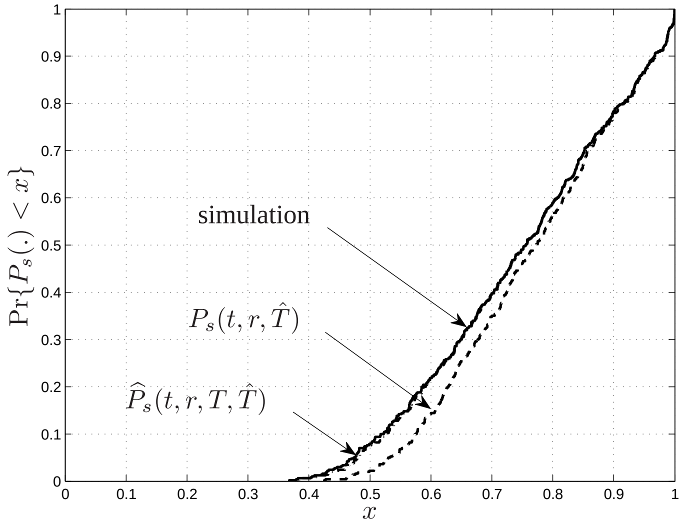

Fig. 3. The cdf of $P_s(t,r,T)$ from simulation (solid line), $P_s(t,r,\hat{T})$ (dashed line), and $\widehat{P}_s(t,r,T,\hat{T})$ (dotted line); $N = 100$, $\lambda = 10$. Note that the dotted line is almost entirely covered by the solid line.

prediction method is more accurate for lower values of Δ.

For the same scenario, the cdfs of $P_s(t,r,T)$ obtained from simulations, $P_s(t,r,\hat{T})$, and $\widehat{P}_s(t,r,T,\hat{T})$ are also shown in Fig. 3. The accuracy of the prediction can be again verified by observing that the cdf of $\widehat{P}_s(t,r,T,\hat{T})$ is almost entirely covered by the cdf of $P_s(t,r,T)$. These results demonstrate a good agreement between the probability of success obtained from the simulation and the prediction method in (32).

VI. Conclusion

We have presented an accurate analysis of the probability of successful transmission of a randomly chosen link in a slotted Aloha network. The proposed method is obtained without any restricting assumptions on the structure of the topology, the traffic model, or transmission power levels. As a matter of fact, the analysis should be applicable for other slotted networks (in addition to slotted Aloha). The flexibility the analysis derives from having considered the detailed network structure and nodes states in our model, which has the negative side effect of a fast growing complexity as the number of nodes in the network increases.

The complexity problem is, however, addressed by proposing an accurate approximation method which initially estimates the link throughput by only considering a small subset of interfering nodes. Furthermore, the error caused by ignoring a set of interfering nodes is predicted and compensated for. A sufficient condition for the accuracy of this prediction method is also presented. The validity of the prediction method is verified for a large number of random networks of 20 and 100 nodes with different node densities. However, a more conclusive evaluation of the accuracy of this approximation method requires more detailed study of different topologies, node densities, and traffic models which is left as a prospect for future work.

Appendix

For a given set of potential interfering node indices T, the disjoint sets of $\hat{T}$ and $\tilde{T}$ are defined according to (14). Also, $\hat{I}$, $\tilde{I}$ are defined according to (15) and (16) respectively. In the following, we prove the validity of the relation used in (22) in which $Q(T,I)$ is replaced with a multiplication of $Q(\hat{T},\hat{I})$ and $Q(\tilde{T},\tilde{I})$.

$$\begin{aligned}
Q\left(T,I\right) &= \prod_{i\in I} p_i \prod_{j\in T\setminus I} \left(1-p_j\right) \\
&= \prod_{i\in \hat{I}\cup\tilde{I}} p_i \prod_{j\in(\hat{T}\setminus\hat{I})\cup(\tilde{T}\setminus\tilde{I})} \left(1-p_j\right) \\
&= \prod_{i\in\hat{I}} p_i \prod_{k\in\tilde{I}} p_k \prod_{j\in\hat{T}\setminus\hat{I}} \left(1-p_j\right) \prod_{h\in\tilde{T}\setminus\tilde{I}} \left(1-p_h\right) \\
&= Q\left(\hat{T},\hat{I}\right) Q\left(\tilde{T},\tilde{I}\right). \qquad (39)
\end{aligned}$$

This result can also be utilized to derive the relation used in (28). Consider a node x such that $x \in I$. Using (39), $Q(T,I)$ can be rewritten as follows

$$\begin{aligned}
Q(T,I) &= Q\left(\{x\},\{x\}\right) Q\left(T\setminus\{x\}, I\setminus\{x\}\right) \\
&= p_x Q\left(T\setminus\{x\}, I\setminus\{x\}\right). \qquad (40)
\end{aligned}$$

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